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INHOMOGENEITIES IN CONSTITUTIVE PROPERTIES AND INITIATION OF SHEAR BANDS IN PLANE STRAIN

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Abstract—Using perturbation analysis, the influence of material inhomogeneities of special geometry on shear band initiation in conditions of plane strain is studied. A hyperelastic material model which represents the behavior of incompressible polycrystallic material is used. Particular attention is given to equivalent strain and pressure distributions. An initiation of main and side shear bands is described and the relation between perturbation analysis results and the phenomena preceding the ductile shear fracture is discussed.

1. INTRODUCTION

Experimental observations suggest that inhomogeneities in constitutive properties play an important role in the initiation and growth of localized shear bands.

The material inhomogeneities may be modelled in several ways. In the basic and most simple variant the inhomogeneity is described as a band (i.e. in a three-dimensional model as a layer) of homogeneous properties [see e.g. Hutchinson and Tvergaard (1981)]. The inhomogeneity in the form of a layer represents an idealized configuration. In actual materials, the presence of approximately equiaxed isolated zones of imperfection is more probable. The strain response of such materials was studied by Abeyaratne and Triantafyllidis (1981). In that study, the authors assumed plane strain deformation and examined the effect of inside homogeneous rectangular inhomogeneity (in plane section) whose geometry was chosen in such a way that the diagonal of the rectangle coincides with the critical direction of shear band initiating in a homogeneous body. Concentration of deformation was shown using the shear component E_{12} of the Lagrangian strain tensor.

The aim of the present study, using the analytic perturbation method of Abeyaratne and Triantafyllidis (1981), is to elucidate the influence of inhomogeneity geometry on shear band initiation, with an emphasis on equivalent strain (ε_e) and pressure (p) distributions. The values of ε_e and p control the processes of strain and damage evolution in real materials—the nucleation and growth of voids—and, finally, the shear fracture. The analytical results also allow for the study of local conditions in the region of the deformation band intersection.

2. HYPERELASTIC MODEL AS A MODEL OF PLASTIC BEHAVIOR OF POLYCRYSTALLIC MATERIAL AT SMALL DEVIATIONS FROM PROPORTIONAL LOADING

As it is commonly known, the deformation theory represents a suitable approximation of polycrystallic material behavior for small deviations from proportional loading [Stören and Rice (1975), Christophersen and Hutchinson (1979)]. More recently, Kitagawa and Matsushita (1987) have compared, using a new approach, the properties of fictitious monocrystal or of polycrystal composed from randomly orientated fictitious monocrystals with the properties of continuum described by the deformation theory. Results of their study also showed that the deformation theory describes the properties of polycrystallic materials in a satisfying manner; the conclusions are not constrained either by a concrete type of crystal lattice with the corresponding geometry of slip systems or by any particular hardening behavior. In the study by Kitagawa and Matsushita (1987) a hypoelastic version of the deformation theory was used [a variant of Stören and Rice (1975)], and the deformations have not exceeded the value of 0.2. Since at higher deformations the difference between the hypo- and hyperelastic version of the deformation theory becomes apparent, it is important to decide which of the two versions is more convenient for modelling polycrystallic material. The experiment can be taken as suitable criterion.

Some results of comparing critical strains ε_c for shear band initiation in hyperelastic material with fracture strains ε_f determined experimentally are presented by Novák (1990), for materials varying significantly in the strain hardening response. With realistic description of strain hardening, the hyperelastic model provides results that correspond well with experimental data. Consequently, we consider this model suitable for analyzing the inhomogeneity influence on shear band initiation.

3. PROBLEM FORMULATION

The method of problem formulation is similar to that used by Abeyaratne and Triantafyllidis (1981) for hyperelastic material. The significant difference lies in the geometry of inhomogeneity and its physical interpretation.

Since many basic relations presented in Abeyaratne and Triantafyllidis (1981) remain valid in our paper (relations (1.1)–(1.11) of their study), we outline only briefly the problem formulation. A hyperelastic body of incompressible material with deformation energy density W is undergoing deformation in such a way that the current point position (y_1, y_2) is related to the initial point position (x_1, x_2) according to the relations:

$$
\begin{aligned}
y_1 &= \lambda_0 x_1 + O(R^{-1}),\\
y_2 &= \lambda_0^{-1} x_2 + O(R^{-1}), \quad \text{as } R = |x| \to \infty,
\end{aligned}
\tag{1}
$$

where λ_0 represents the applied stretch at infinity in the x_1-direction. Under the plane strain conditions and considering material incompressibility, the potential W may be expressed as a function of the invariant, I, of the right Cauchy Green tensor, C, and of the initial position of the material point $x = (x_1, x_2)$, $W = W(I, x)$. We assume that the inhomogeneity is in the plane section limited to a bounded region D. That is, function W is of the form:

$$
W(I, x) = W_0(I) + \xi \tilde{W}(I, x) \tag{2}
$$

where W_0 is the elastic potential associated with the homogeneous body, ξ is the "measure of imperfection" and $\tilde{W}$ is a function whose support is equal to D, i.e.

$$
\begin{aligned}
\tilde{W}(I, x) &= 0, \quad \text{for } x \in R^2 - D,\\
\tilde{W}(I, x) &\neq 0, \quad \text{for } x \in D.
\end{aligned}
$$

If we consider a simple power hardening law with exponent $1/n$, then $W_0(I)$ may be written in the form:

$$
W_0(I) = [n/(n+1)] \cdot k \cdot \{0,5 \cdot \ln[(I + \sqrt{(I^2 - 4)})/2]\}^{(n+1)/n}. \tag{3}
$$

In (2), we suppose that $\tilde{W}(I, x) = \tilde{W}(I^0, x)$ holds, where I^0 is the unperturbed part of I associated with the homogeneous body. The function $\tilde{W}(I^0, x)$ is assumed to be in the form:

$$
\tilde{W}(I^0, x) = W_0(I^0) \cdot f(x_1, x_2), \tag{4}
$$

where $f = f(x_1, x_2)$ will be specified in the following text. The form (4) is in accordance with the asumption that the inhomogeneity material differs from material outside the inhomogeneity only in the value of the strength coefficient, whilst preserving the qualitative character of stress–strain relations. In our case, this means that values of the strain hardening exponent are the same in both parts of the body. We assume the inhomogeneity to have a circular cross-section in its undeformed state, in particular, numerical calculations were performed under the assumptions ($a > 0$):

$$f(x_1, x_2) = \cos[(\pi/(2a))\sqrt{(x_1^2+x_2^2)}], \quad \text{for } \sqrt{(x_1^2+x_2^2)} \leqslant a,$$
$$f(x_1, x_2) = 0, \qquad \text{for } \sqrt{(x_1^2+x_2^2)} > a. \tag{5}$$

The consequences following from the form of $f(x_1, x_2)$ will be discussed qualitatively further in the text.

Under these assumptions we look for a space distribution of certain quantities that may be derived from $y_z(x)$ and $p(x)$, where p is a scalar pressure field. Using the equilibrium equations and the incompressibility condition, we obtain a system of equations for the unknown functions $y_z(x)$ and $p(x)$.

4. PERTURBATION ANALYSIS

Functions y_1, y_2 and p are assumed to be functions of x and ξ, for $\xi \to 0$ we may write:

$$\begin{aligned} y_1 &= \lambda_0 x_1 + \xi \bar{u}_1(x) + \sigma(\xi), \\ y_2 &= \lambda_0^{-1} x_2 + \xi \bar{u}_2(x) + \sigma(\xi), \\ p &= p_0 + \xi \bar{p}(x) + \sigma(\xi). \end{aligned} \tag{6}$$

To satisfy the incompressibility condition we use the displacement potential $\psi(x)$ such that

$$\bar{u}_1 = \lambda_0(\partial\psi/\partial x_2), \quad \bar{u}_2 = -\lambda_0^{-1}(\partial\psi/\partial x_1). \tag{7}$$

After the elimination of p from the equilibrium equations we obtain

$$(\mu - \sigma/2)\psi_{,1111} + 2(2\mu^* - \mu)\psi_{,1122} + (\mu + \sigma/2)\psi_{,2222} = g, \tag{8}$$

where μ, σ and μ^* have the usual meaning, i.e the same meaning as in the linearized form of the constitutive equations for plane strain tension for isotropic, incompressible, elastic material under the conditions of special axes' orientation (Hill and Hutchinson, 1975):

$$\overset{\vee}{\sigma}_{11} - \overset{\vee}{\sigma}_{22} = \overset{\vee}{\sigma} = 2\mu^*(\dot{\varepsilon}_{11} - \dot{\varepsilon}_{22})$$
$$\overset{\vee}{\sigma}_{12} = \mu\dot{\varepsilon}_{12}.$$

Function g can be expressed using eqn (4) as

$$g = -\sigma\,\partial^2 f(x_1, x_2)/\partial x_1\,\partial x_2.$$

If $2\mu - \mu^* + (\mu^2 - \sigma^2/4)^{0.5} > 0$ (the strain level is lower than the critical strain for shear band initiation in the perfect body) then the solution of eqn (8) may be written in the form (Kofroň *et al.*, 1988):

$$\psi(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_1 - \omega_1, x_2 - \omega_2) \cdot g(\omega_1, \omega_2)\, d\omega_1\, d\omega_2, \tag{9}$$

where

$$G(x_1, x_2) = \frac{1}{(4\pi)^2} \int_0^{2\pi} \frac{(x_1 \cos\phi + x_2 \sin\phi)^2 . \ln(x_1 \cos\phi + x_2 \sin\phi)^2}{(\mu - \sigma/2)\cos^4\phi + 2(2\mu^* - \mu)\cos^2\phi \sin^2\phi + (\mu + \sigma/2)\sin^4\phi} \, d\phi.$$

The solution was found using the theory of distributions (Shilov, 1965). The function $G(x_1, x_2)$ in (9) may be explicitly expressed in terms of elementary functions, using the methods of complex analysis (Kofroň *et al.*, 1989).

5. EVALUATION OF STRAIN AND PRESSURE FIELDS

The displacement potential $\psi(x_1, x_2)$ allows for the possibility of deriving, in a simple manner, two characteristics of the strain field: the equivalent strain ε_e and the shear component E_{12} of the Lagrangian strain tensor E. Similarly, as in Abeyaratne and Triantaphyllidis (1981), we obtain:

$$\begin{aligned} \varepsilon_e &= \varepsilon_e^0 + \xi\bar{\varepsilon}_e = (2/3^{0.5}) \cdot \ln \lambda_0 (2/3^{0.5})\xi\psi_{,12}, \\ E_{12} &= E_{12}^0 + \xi\bar{E}_{12} = 0 + (1/2)\xi(\lambda_0^2\psi_{,22} - \lambda_0^{-2}\psi_{,11}). \end{aligned} \tag{10}$$

In order to determine the factor $\bar{p}$ in eqn (6) we must return to the equilibrium equations. It is not difficult to show (Novák and Lauerová, 1990) that for the region outside the inhomogeneity the following equation holds:

$$\bar{p} = \lambda_0^2/(\lambda_0^2 - \lambda_0^{-2})\left\{\sigma \int \psi_{,222}\, dx_1 + [4(\mu^* - \mu) + \sigma]\psi_{,12}\right\}. \tag{11}$$

The expressions for $\psi_{,12}$, $\psi_{,22}$, $\psi_{,11}$ and $\int \psi_{,222}\, dx_1$ which are met in eqns (10) and (11) may be expressed in the form:

$$\begin{aligned} \psi_{,\alpha\beta} &= \int_{-a}^{a} \int_{-a}^{a} \partial^2 G(x_1 - \omega_1, x_2 - \omega_2)/\partial x_\alpha \partial x_\beta \cdot g(\omega_1, \omega_2)\, d\omega_1\, d\omega_2, \\ \int \psi_{,222}\, dx_1 &= \int_{-a}^{a} \int_{-a}^{a} \Phi(x_1, x_2; \omega_1, \omega_2) \cdot g(\omega_1, \omega_2)\, d\omega_1\, d\omega_2, \end{aligned} \tag{12}$$

where

$$\Phi(x_1, x_2; \omega_1, \omega_2) = \int \partial^3 G(x_1 - \omega_1, x_2 - \omega_2)/\partial x_2^3 \cdot dx_1.$$

The integration in (12) represents an integration over the circle domain $\sqrt{(\omega_1^2 + \omega_2^2)} \leqslant a$ with respect to (5). Explicit expressions for partial derivatives of function G were presented by Novák and Lauerová (1990).

Remarks on numerical calculations

We used the following integration formula. It represents a modification of a basic formula presented in Krylov (1967):

$$\iint_\Omega F(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 = (3\pi/20) \cdot (5/n) \sum_{k=0}^{n-1} F[(2/3)^{0.5} \cos(2k\pi/n + \phi), (2/3)^{0.5} \sin(2k\pi/n + \phi)], \tag{13}$$

where Ω is a circle $\sqrt{(x_1^2 + x_2^2)} \leqslant a$. We took $n = 50$ in (13).

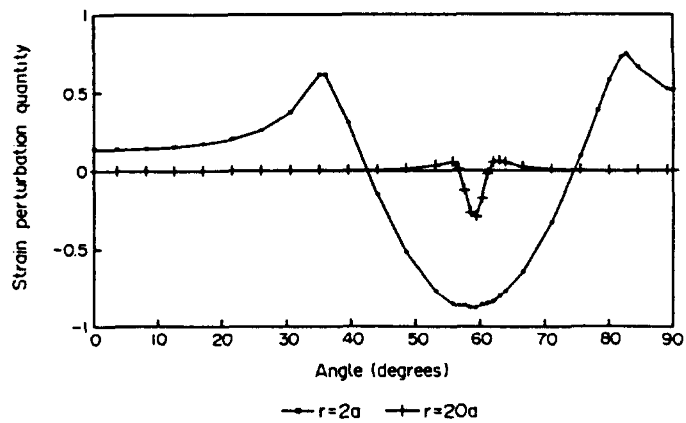

Fig. 1. Angular variation of $\bar{\varepsilon}_e$ along the circular arcs of different radii. Strain hardening exponent $1/n = 0.25$, deformation measure $l = (\lambda_0 - 1)/(\lambda_c - 1) = 0.98$.

6. RESULTS OF CALCULATIONS

6.1. *The distributions of selected quantities in the vicinity of inhomogeneity*

Figures 1, 2 show angular variations in the quantities $\bar{\varepsilon}_e$ and $\bar{p}$ along circular arcs with radii $r = 2a$ and $r = 20a$, placed in the center of the inhomogeneity domain, for $n = 4$ and $l = (\lambda_0 - 1)/(\lambda_c - 1) = 0.98$. The characteristic angle in the undeformed configuration is $\theta_c = 59,3°$. Both quantities take maximum absolute values in the direction corresponding approximately to the characteristic angle θ_c and determine the local conditions in the main shear band. Other local peaks encountered in Figs 1 and 2 correspond to side deformation bands of opposite signs. According to Abeyaratne and Triantafyllidis (1981), the existence of three peaks in an angular variation of $\bar{E}_{12}$ is associated with three characteristics passing through the vertices of the inhomogeneity. (The inhomogeneity was assumed in the form of a rectangle whose diagonal is inclined from the x_1-axis at the characteristic angle θ_c.) Since the three peaks in angular variations of quantities $\bar{E}_{12}$, $\bar{\varepsilon}_e$ and $\bar{p}$ have also appeared for circular inhomogeneity, we are of the opinion that the reason for the side band initiation lies in the geometrical constraints of the problem. This reason may be elucidated comparing two models containing different types of inhomogeneity. In the first model (Hutchinson and Tvergaard, 1981) the inhomogeneity is supposed in the form of a layer and a free relative displacement of two homogeneous parts of the body (outside the inhomogeneity) is possible, in consequence of which the side bands do not appear. In contrast to this, in Abeyaratne and Triantafyllidis' (1981) model (and also in our own model) such a displacement is excluded and the side deformation bands compensate for the effect of the main deformation band.

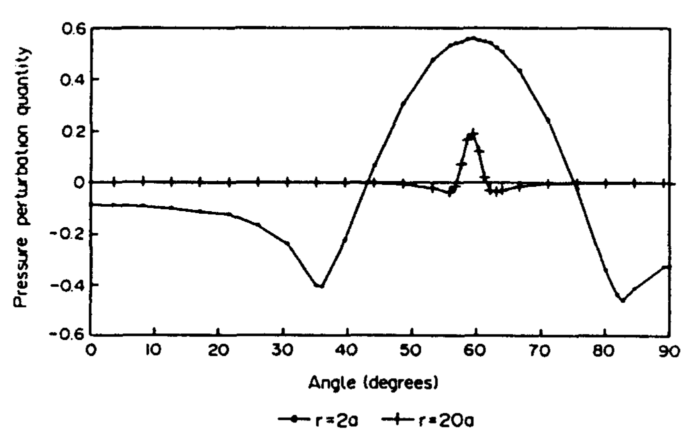

Fig. 2. Angular variation of $\bar{p}$ along the circular arcs of different radii. Strain hardening exponent $1/n = 0.25$, deformation measure $l = (\lambda_0 - 1)/(\lambda_c - 1) = 0.98$.

Angular variations in the quantities $\bar{E}_{12}$ and $\bar{\varepsilon}_e$ have similar shapes in the peak region, but they differ near the problem symmetry axes. ($\bar{E}_{12}$ is equal to zero on these axes.) Quantities $\bar{\varepsilon}_e$ and $\bar{p}$ attain non-zero values on these axes, and, moreover, they are approximately proportional to each other.

An interesting effect of the rotationally symmetric inhomogeneity profile may be shown. The profile corresponding to the function $f = -|x|^n + a^n$, $x = (x_1, x_2)$, $|x| < a$, may be taken as another possibility comparable with the cosine profile described by (5). Considering various functions, f, which differ in exponent n, the deformation bands change their signs in going from $n < 2$ to $n > 2$ or inversely. For $n = 2$ the inhomogeneity initiates no deformation bands. This effect also explains why the deformation bands associated with the cosine profile are of the opposite signs to those described by Abeyaratne and Triantafyllidis (1981).

Using an analytic formula it can be shown which effects arise in the case of shear band intersection. If we consider two inhomogeneities (of the same type and size) with the distance between the centers being greater than $2a$, it follows from eqn (12) that the values of $\bar{\varepsilon}_e$, $\bar{p}$ and $\bar{E}_{12}$ obtained are equal to the sum of the relevant values associated with each of the inhomogeneities. In particular, the values of the shear strain component, $\bar{E}_{12}$, vanish in the region of band intersection.

6.2. *Influence of both the straining level and the strain hardening exponent on the behavior of selected quantities near the inhomogeneity*

As is known from the results of shear band bifurcation analysis, the change in the strain hardening exponent $1/n$ results in the change of both critical strain for shear band initiation and band orientation. The distributions of $\bar{\varepsilon}_e$ and $\bar{p}$ display similar characteristics for the strain level close to the critical strain, for both $n = 10$ and $n = 4$. The calculations of $\bar{\varepsilon}_e$ and $\bar{p}$ were performed for $n = 4$ and $n = 10$, for strain levels characterized by different values of l, $l = 0.94$, $l = 0.96$ and $l = 0.98$. Figures 3 and 4 show the angular variation of $\bar{\varepsilon}_e$ along the circular arc with radius $r = 20a$. It is seen that $\bar{\varepsilon}_e$ is concentrated highly in the region of the main deformation band ($\bar{\varepsilon}_e$ changes approximately proportionally to $(1-l)^{-1}$).

7. DISCUSSION

The relationship between model calculation results and experimental observations of deformation bands has already been discussed by Abeyaratne and Triantafyllidis (1981). The results of the present work permit certain conclusions:

The main deformation bands are accompanied by side deformation bands of opposite signs, therefore deformation bands may be initiated by both negative and positive inhom-

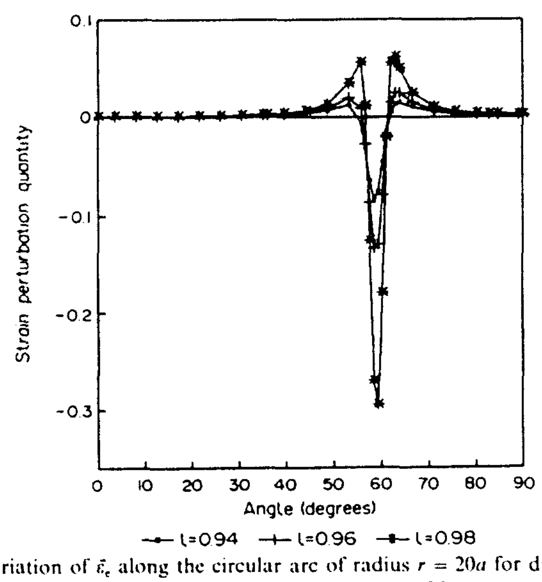

Fig. 3. Angular variation of $\bar{\varepsilon}_e$ along the circular arc of radius $r = 20a$ for different strain levels. Strain hardening exponent $1/n = 0.25$.

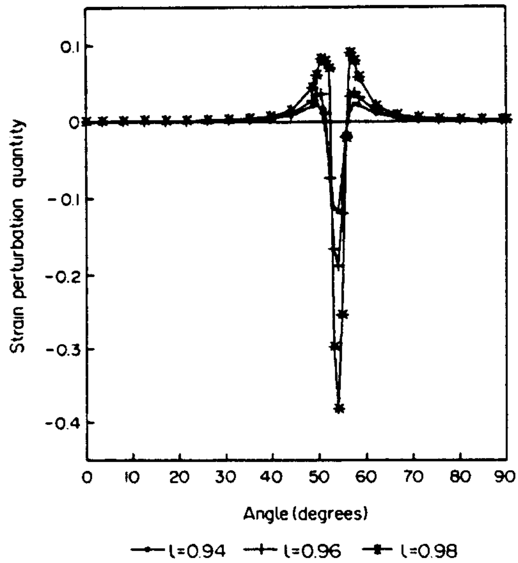

Fig. 4. Angular variation of $\bar{\varepsilon}_e$ along the circular arc of radius $r = 20a$ for different strain levels. Strain hardening exponent $1/n = 0.1$.

ogeneities. (Positive inhomogeneity may be represented by a region with a greater concentration of rigid inclusions.) This is the difference from the layer model presented by Hutchinson and Tvergaard (1981) where the positive inhomogeneity does not promote the strain concentration and localization. Moreover, a change in inhomogeneity profile can cause the change in the signs of the main and side deformation bands.

The hydrostatic pressure difference develops in proportion to the difference in equivalent strain. The same relation holds for an inhomogeneity layer model (Novák and Lauerová, 1990). Consequently, we may describe cavity nucleation as a part of the fracture process by using unique local parameter—local deformation, even though nucleation depends in general on both parameters—local deformation and local hydrostatic pressure.

The additivity of $\bar{\varepsilon}_e$ and $\bar{p}$ in the region of the deformation band intersection corresponds to experience—the initiation of the macroscopic crack proceeds in the region of the deformation band intersection. The initiation of the deformation bands may be identified with the initiation of fracture; the knowledge of local conditions in the region of the deformation band intersection permits better understanding of the phenomena preceding the fracture.

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